Math 304 Midterm 2 Sample

Name: ____________________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	12	
2	5	
3	10	
4	10	
5	15	
6	12	
7	10	
8	16	
9	10	
Total:	100	

Question 1. (12 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) If A is an $m \times n$ matrix, then A and A^T have the same rank.

(b) Given two matrices A and B, if B is row equivalent to A, then B and A have the same row space.

(c) Given two vector spaces, suppose $L : V \to W$ is a linear transformation. If S is a subspace of V, then $L(S)$ is a subspace of W.

(d) For a homogeneous system of rank r and with n unknowns, the dimension of the solution space is $n - r$.

Question 2. (5 pts)

Find the angle between $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$.

Question 3. (10 pts)

Let V be the subspace of $\mathbb{R}^3$ spanned by $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Describe $V^{\perp}$ by finding a basis of $V^{\perp}$.

Question 4. (10 pts)

Given two lines

$$L_1 : x = t + 1,\ y = 3t + 1,\ z = 2t - 1,$$

and

$$L_2 : x = 2t - 2,\ y = 2t + 3,\ z = t + 1,$$

suppose a plane H is parallel to both L_1 and L_2. Moreover, H passes through the point $(0, 1, 0)$. Find an equation of H.

Question 5. (15 pts)

Given

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

(a) Find a basis of $\mathrm{Ker}(A)$.

(b) Find a basis of the row space of A.

(c) Find a basis of the range of A.

(d) Determine the rank of A.

Question 6. (12 pts)

Determine whether the following mappings are linear transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 1 \\ x_1 + x_2 \end{pmatrix}$$

(b) $L : \mathbb{P}_2 \to \mathbb{P}_2$ by

$$L(p(x)) = p'(x) + p(x)$$

Question 7. (10 pts)

Let $M_2(\mathbb{R})$ be the space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the basis S.

Question 8. (16 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 - x_2 \end{pmatrix}$$

(a) Find the matrix representation of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

(b) Let $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Moreover, let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Find the matrix representation of L with respect to the basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

Question 9. (10 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Its matrix representation with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix}.$$

(a) Find the transition matrix from the basis $\{u_1, u_2\}$ to the standard basis $\{e_1, e_2\}$, where

$$u_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(b) Find the matrix representation of L with respect to $\{u_1, u_2\}$.